Ultrametricity in the Kauffman model: a numerical test

1988 J. Phys. A: Math. Gen. 21 L357

(http://iopscience.iop.org/0305-4470/21/6/007)

J. Phys. A: Math. Gen. **21** (1988) L357–L361. Printed in the UK

LETTER TO THE EDITOR

Ultrametricity in the Kauffman model: a numerical test

E N Miranda and N Parga

Centro Atómico Bariloche† and Instituto Balseiro‡, 8400 S C de Bariloche, Argentina

Received 23 December 1987

Abstract. We perform a numerical test to check the existence of ultrametricity in the set of cycles of random Boolean automata networks. We find signs of it when the test is applied to networks with connectivity two and three. However, our system sizes are probably still too small to conclude whether it is trivial (only equilateral triangles) or not.

In 1969, Kauffman (1969) proposed a model to explain some aspects of cell differentiation. It consists of a random Boolean automata network. The question is to determine whether there exist properties which are shared by almost all the nets, with possibly few exceptions corresponding to special samples.

Recently, some attention has been paid to this model because its configuration space seems to be like that of spin glasses. Indeed, the use of spin-glass concepts in the study of random automata has been fruitful (Derrida and Flyvbjerg 1986, Miranda and Parga 1987). Some relevant magnitudes of mean-field spin-glass theory have been redefined in the Kauffman model and it has been shown that they obey similar, but not identical (Derrida and Flyvbjerg 1987), relations in both models. In this letter, we continue the study of the Kauffman model in this direction. Our aim is to test ultrametricity in the space of cycles weighed by their attraction basins. In order to check this property we define a distance between cycles and perform a numerical test.

After reviewing the Kauffman model and the mathematical concept of ultrametricity, we define the distance between cycles and check that at least for large N the definition is a sensible one. Next we evaluate the cycle overlap distribution and perform a numerical test of ultrametricity.

Consider a system of N automata $S_i = 0, 1$. For each node S_i one gives k input sites $j_1(i), \cdots, j_b(i)$ and a Boolean function f_i of k variables. Then the dynamics of this system is defined by:

$$S_i(t+1) = f_i(S_{j_1(i)}(t) \ldots S_{j_k(i)}(t)).$$

The Boolean functions are completely characterised by 2^k values. The functions f_i and the inputs $j_1(i), \ldots, j_k(i)$ of each site i are chosen at time $t = 0$ and remain fixed for the rest of the system evolution. This implies a deterministic dynamics: the system configuration at time $t+1$ is completely given by that at time t; and since the configuration space is finite the evolution ends up by becoming periodic for any initial configuration. There are in general several periodic attractors and the phase space is broken according to their attraction basins. These have been studied recently by Derrida and Flyvbjerg (1986) and ourselves (Miranda and Parga 1987). The multivalley structure is qualitatively similar to that of the mean-field spin glass (Derrida and Flyvbjerg 1986) and surprising analogies between the two systems have been found.

† Comisión Nacional de Energía Atómica, Argentina.

‡ Comisión Nacional de Energía Atómica and Universidad Nacional de Cuyo, Mendoza, Argentina.

0305/4470/88/060357+05$2.50

Ultrametricity is a simple topological concept but its appearance in physics is recent (Rammal *et al* 1985, 1986, Parga 1987). It is also found in other fields of knowledge such as optimisation theory and biology. From a mathematical point of view, an ultrametric space is characterised by the following inequality:

$$d[A, C] \leq \max\{d[A, B], d[B, C]\} \tag{1}$$

where A, B, C are points in that space and $d[x, y]$ is a properly defined distance on it. One should remember that in an ordinary metric space the following relations hold:

$$d[x, y] = 0 \qquad \text{if and only if } x = y \tag{2a}$$

$$d[x, y] = d[y, x] \tag{2b}$$

$$d[x, z] \leq d(x, y) + d(y, z). \tag{2c}$$

Thus, in an ultrametric space, there is a stronger constraint on distances: relation (1) must hold instead of the weaker condition (2c). This implies that in an ultrametric space all the triangles are either equilateral or isosceles with the shortest side as the base.

There are analytical results which suggest that the Kauffman model is ultrametric. Derrida and Pomeau (1986) have proposed an annealed approximation to the model. Numerical simulations show that this approximation works well also in the quenched model (Derrida and Weisbuch 1986). Finally, Hilhorst and Nijmeijer (1987) have been able to prove that for some quantities there is no difference between the annealed approximation and the quenched model in the limit $N \to \infty$: if one chooses a couple of initial configurations and lets them evolve for a long time, their final average distance converges to a value $d(k, N)$ independent of the initial states. This means that if one chooses three random initial configurations it is likely that their final distances form an equilateral triangle. This result tells us little about the cycle space topology but it suggests ultrametricity may be present in the Kauffman model.

We define the distance between two cycles A and B as the average Hamming distance between all possible pairs of configurations, taken one in cycle A and the other in cycle B. It is not straightforward that this distance verifies all the relations (2). Obviously, condition (2b) holds trivially but (2a) becomes true only as $N \to \infty$. We calculated $d[x, x]$ numerically, and found its probability distribution has a peak at zero, with its value increasing with N. There are also some triplets that do not verify the inequality (2c) but they disappear quickly with increasing N.

Next we evaluate the probability distribution $P(q)$ of the overlap $q = 1 - d$, between pairs of cycles. Typically 10 000 samples were analysed for $k = 2$ and $k = 3$ and several values of N ranging from 10 to 128 in the first case and from 10 to 64 in the second one. Figures (1a) and (1b) show $P(q)$ for $k = 2$, $N = 48$ and for $k = 3$, $N = 32$ respectively. We have found that $P(q)$ for $k = 3$ takes a shape which for those values of N does not show any appreciable dependence on N. However, for $k = 2$ the distribution parameters vary with N. We have found that for small $N = 10$–20, then q^*—the value for which $P(q)$ reaches its maximum—is about 0.7 while for larger $N = 96$–128 it shifts to $q^* = 0.9$. On the other hand for $k = 3$ it stays at $q^* = 0.62$. It is clear that for $k = 2$, $q^* \to 1$. These values are in agreement with those found by Derrida and Pomeau (1986): $q^* = 1$ for $k = 2$ and $q^* = 0.628$ for $k = 3$. Our numerical results show that the distribution width does not decrease with increasing N. As an example, $P(q)$ for $N = 96$ and $k = 2$ can be seen in figure 2. This suggests that $P(q)$ does not become a delta function as $N \to \infty$. However, one should be cautious since probably our values of N are still too small to make a conclusive statement about this

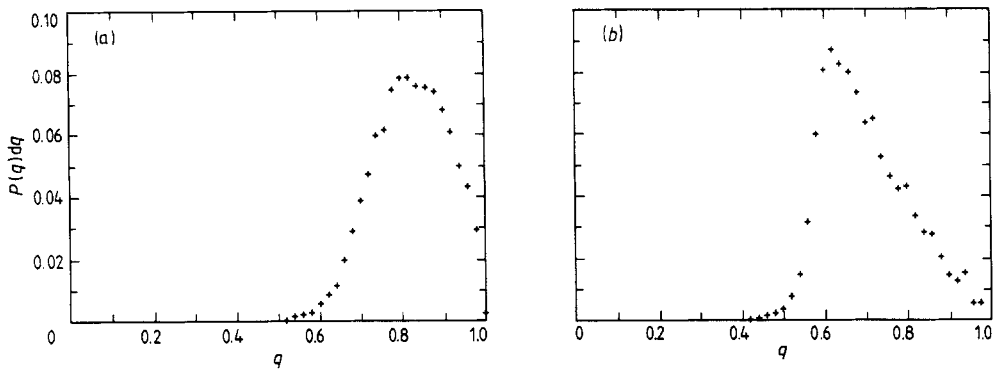

Figure 1. The overlap distribution $P(q)$ for: (*a*) $k=2$ and $N=48$; (*b*) $k=3$ and $N=32$.

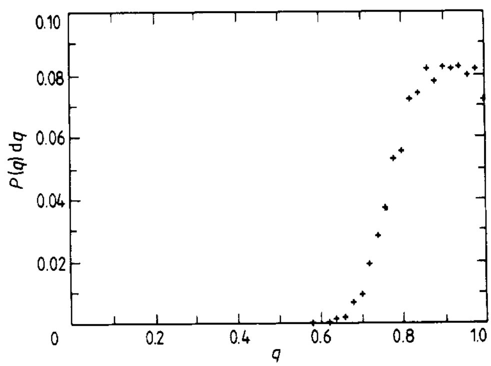

Figure 2. The overlap distribution $P(q)$ for $k=2$ and $N=96$. The distribution width does not decrease with increasing N (cf figure 1(*a*)).

point. In the case of the travelling salesman problem it became clear that $P(q)$ was a delta function only when larger values of N and good quasi-optimal tours were considered (Sourlas 1986).

In order to check if ultrametricity holds we propose the following numerical test, which resembles that used by Bhatt and Young (1986) for spin glasses: a great number of triangles with a fixed $d_{\min} = 1 - q_{\max}$ are studied, where $d_{\min}$ is the size of the smallest side of each one. The overlap difference q of the other two sides is measured for each triangle:

$$\delta q = q_{\mathrm{med}} - q_{\min}$$

and the probability distribution $\phi(\delta q)$ is obtained. If ultrametricity were exact this distribution should approach a delta function at zero as $N \to \infty$. For finite N the measured distribution appears spread around the origin due to finite-size effects.

Figures (3*a*) and (3*b*) show typical $\phi(\delta q)$ histograms for $k=2$ and $N=15$ and 128 respectively. Figures (4*a*) and (4*b*) show the same distribution for $k=3$ and $N=20$ and $N=64$ respectively. It is clear that the distribution peak in zero increases with N. In table 1, peak values $\phi(0)$ are shown for different values of N and k. These numerical results are a good indication of the existence of ultrametricity between cycles.

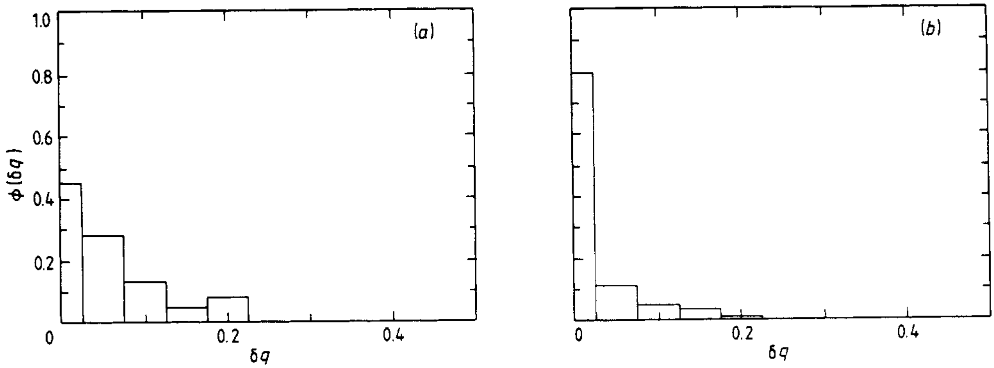

Figure 3. Two typical examples of distribution $\phi(\delta q)$, for: (*a*) $k=2$, $N=15$; (*b*) $k=2$, $N=128$. In all cases, q_{max} was chosen equal to 0.8 for $k=2$. The histogram step has been chosen equal to 0.05

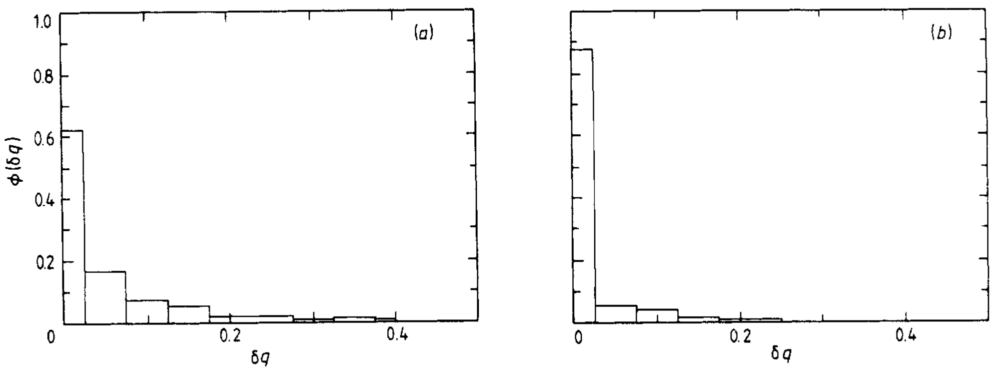

Figure 4. Distribution $\phi(\delta q)$ for: (*a*) $k=3$ and $N=20$; (*b*) $k=3$ and $N=64$. q_{max} was chosen equal to 0.6 for $k=3$. The histogram step has been chosen equal to 0.05.

Table 1. Peak $\phi(0)$ for different k and N. In this case, the histogram step has been chosen equal to 0.1.

	N							
	10	15	20	32	48	64	96	128
$K=2$	0.57	0.76	0.79	0.84	0.89	0.89	0.90	0.91
$K=3$	0.53	0.64	0.70	0.78	0.90	0.93	—	—

To conclude let us notice that the only crucial difference between the cases $k=2$ and $k=3$ is in the value of q^*. While for $k=2$ it approaches 1 as N increases, for $k=3$ it reaches a constant. However, the corresponding widths of $P(q)$ do not seem to become smaller for the system sizes we studied. This seems to be in contradiction with the existence of different phases for $k=2$ and $k=3$. For $k=2$ only macroscopically equal configurations are obtained as the evolution proceeds to large times (Derrida and Weisbuch 1986, Hilhorst and Nijmeijer 1987). Apart from finite-size effects this difference might be due to a different definition of overlaps. In the above references the overlaps were obtained as the average over initial conditions and several samples and the time evolution was stopped without checking whether a cycle had been reached or not. Besides, the width of the distribution was not obtained for the quenched version of the Kauffman model. It is not clear to us that the average over pairs of initial configurations falling in the same pair of attractors gives the same overlap as the one defined by us. The existence of privileged points to enter the cycles would imply a difference in the overlaps.

One of us (ENM) would like to thank Miguel A Virasoro for discussions at the beginning of this work.

References

Bhatt R N and Young A P 1986 *J. Magn. Magnet. Mater.* **54-57** 191
Derrida B and Flyvbjerg H 1986 *J. Phys. A: Math. Gen.* **19** L1003
—— 1987 *J. Physique* **48** 971
Derrida B and Pommeau Y 1986 *Europhys. Lett.* **1** 45
Derrida B and Weisbuch G 1986 *J. Physique* **47** 1297
Hilhorst H J and Nijmeijer M 1987 *J. Physique* **48** 185
Kauffman S A 1969 *J. Theor. Biol.* **22** 437
Miranda E N and Parga N 1987 *Europhys. Lett.* submitted
Parga N 1987 *J. Physique* **48** 499
Rammal R, Angles D'Auriac J C and Duçot B 1985 *J. Physique Lett.* **46** L223
Rammal R, Toulouse G and Virasoro M A 1986 *Rev. Mod. Phys.* **58** 765
Sourlas N 1986 *Europhys. Lett.* **2** 919